# Absence of exponentially localized solitons for the Novikov–Veselov equation at negative energy

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**Abstract.** We show that Novikov–Veselov equation (an analog of KdV in dimension $2+1$) does not have exponentially localized solitons at negative energy.

# 1 Introduction

In the present paper we are concerned with the following $(2+1)$–dimensional analog of the Korteweg–de Vries equation:

$$\begin{aligned}
&\partial_t v = 4\mathrm{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w),\\
&\partial_{\bar z} w = -3\partial_z w, \quad v = \bar v, \quad E \in \mathbb{R},\\
&v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R},
\end{aligned} \tag{1.1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right). \tag{1.2}$$

Equation (1.1) is contained implicitly in the paper of S.V. Manakov [M] as an equation possessing the following representation

$$\frac{\partial(L-E)}{\partial t} = [L-E, A] + B(L-E) \tag{1.3}$$

(Manakov L–A–B triple), where $L = -\Delta + v(x,t)$, $\Delta = 4\partial_z\partial_{\bar z}$, $A$ and $B$ are suitable differential operators of the third and zero order respectively, $[\cdot,\cdot]$ denotes the commutator. Equation (1.1) was written in an explicit form by S.P. Novikov and A.P. Veselov in [NV1], [NV2], where higher analogs of (1.1) were also constructed. Note that both Kadomtsev–Petviashvili equations can be obtained from (1.1) by considering an appropriate limit $E \to \pm\infty$ (see [ZS], [G]).

In the case when $v(x_1, x_2, t)$, $w(x_1, x_2, t)$ are independent of $x_2$, (1.1) can be reduced to the classic KdV equation:

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{1.4}$$

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It is well–known that (1.4) has the soliton solutions

$$u(x,t) = u_{\kappa,\varphi}(x-4\kappa^2 t) = -\frac{2\kappa^2}{ch^2(\kappa(x-4\kappa^2 t-\varphi))}, \quad x \in \mathbb{R},\, t \in \mathbb{R},\, \kappa \in (0,+\infty),\, \varphi \in \mathbb{R}. \tag{1.5}$$

Evidently,

$$\begin{aligned} &u_{\kappa,\varphi} \in C^\infty(\mathbb{R}),\\ &\partial_x^j u_{\kappa,\varphi}(x) = O(e^{-2\kappa|x|}) \text{ as } x \to \infty, \quad j = 0,1,2,\ldots \end{aligned} \tag{1.6}$$

Properties (1.6) imply that the solitons (1.5) are exponentially localized in $x$.

For the 2–dimensional case we will say that a solution $(v,w)$ of (1.1) is an exponentially localized soliton if the following properties hold:

$$\begin{aligned} &v(x,t) = V(x-ct), \quad x \in \mathbb{R}^2, \quad c = (c_1,c_2) \in \mathbb{R}^2,\\ &V \in C^3(\mathbb{R}^2), \quad \partial_x^j V(x) = O(e^{-\alpha|x|}) \text{ for } |x| \to \infty, \quad |j| \leqslant 3 \text{ and some } \alpha > 0\\ &(\text{where } j = (j_1,j_2) \in (0 \cup \mathbb{N})^2, |j| = |j_1| + |j_2|, \quad \partial_x^j = \partial^{j_1+j_2}/\partial x_1^{j_1}\partial x_2^{j_2}),\\ &w(\cdot,t) \in C(\mathbb{R}^2), \quad w(x,t) \to 0 \text{ as } |x| \to \infty, \quad t \in \mathbb{R}. \end{aligned} \tag{1.7}$$

In [N1] it was shown that, in contrast with the $(1+1)$–dimensional case, the $(2+1)$–dimensional KdV equation (1.1), at least for $E = E_{fixed} > 0$, does not have exponentially localized solitons. More precisely, in [N1] it was shown that the following theorem is valid for $E = E_{fixed} > 0$:

**Theorem 1.1.** *Let $(v,w)$ be an exponentially localized soliton solution of (1.1) in the sense (1.7). Then $v \equiv 0$, $w \equiv 0$.*

The main result of this paper consists in the proof of Theorem 1.1 for the case $E = E_{fixed} < 0$. This proof is given in Section 3 and is based on Propositions 3.1 and 3.2. In addition: Proposition 3.1 is an analog of the result of [N1] about the transparency of sufficiently localized solitons for equation (1.1) for $E > 0$; Proposition 3.2 is an analog of the result of [N2], [GN1] that there are no nonzero bounded real–valued exponentially localized transparent potentials (that is potentials with zero scattering amplitude) for the Schrödinger equation (2.1) for $E = E_{fixed} > 0$.

Note that nonzero bounded algebraically localized solitons for equation (1.1) for $E < 0$ are also unknown (see [G]), but their absence is not proved.

As regards integrable systems in $2+1$ dimensions admitting exponentially decaying solitons in all directions on the plane, see [BLMP], [FS].

As regards integrable systems in $2+1$ dimensions admitting nonzero bounded algebraically decaying solitons in all directions on the plane, see [FA], [BLMP], [G], [KN] and references therein.

## 2 Inverse scattering for the 2–dimensional Schrödinger equation at a fixed negative energy

Consider the scattering problem for the two–dimensional Schrödinger equation at a fixed negative energy:

$$\begin{aligned} &-\Delta\psi + v(z)\psi = E\psi, \quad E = E_{fixed} < 0, \\ &\Delta = 4\partial_z\partial_{\bar z}, \quad z = x_1 + ix_2, \quad x \in \mathbb{R}^2, \end{aligned} \tag{2.1}$$

where $\partial_z$, $\partial_{\bar z}$ are the same as in (1.2). We will assume that the potential $v(z)$ satisfies the following conditions

$$\begin{aligned} &v(z) = \overline{v(z)}, \quad v(z) \in L^\infty(\mathbb{C}), \\ &|v(z)| < q(1+|z|)^{-2-\varepsilon} \text{ for some } q > 0,\ \varepsilon > 0. \end{aligned} \tag{2.2}$$

In this paper we will be concerned with the exponentially decreasing potentials, i.e. with the potentials $v(z)$ satisfying (2.2) and the following additional condition

$$v(z) = O(e^{-\alpha|z|}) \text{ as } |z| \to \infty \text{ for some } \alpha > 0. \tag{2.3}$$

Direct and inverse scattering for the two–dimensional Schrödinger equation (2.1) at fixed negative energy under assumptions (2.2) was considered for the first time in [GN2]. For some of the results discussed in this section see also [N2], [G].

First of all, we note that by scaling transform we can reduce the scattering problem with an arbitrary fixed negative energy to the case when $E = -1$. Therefore, in our further reasoning we will assume that $E = -1$.

It is known that for $\lambda \in \mathbb{C}\backslash(0 \cup \mathcal{E})$, where

$$\begin{aligned} &\mathcal{E} \text{ is the set of zeros of the modified Fredholm determinant } \Delta \\ &\text{for the integral equation (2.10),} \end{aligned} \tag{2.4}$$

there exists a unique continuous solution $\psi(z,\lambda)$ of (2.1) with the following asymptotics

$$\psi(z,\lambda) = e^{-\frac{1}{2}(\lambda\bar z + z/\lambda)}\mu(z,\lambda), \quad \mu(z,\lambda) = 1 + o(1), \quad |z| \to \infty. \tag{2.5}$$

In addition, the function $\mu(z,\lambda)$ satisfies the following integral equation

$$\mu(z,\lambda) = 1 + \iint\limits_{\zeta\in\mathbb{C}} g(z-\zeta,\lambda)v(\zeta)\mu(\zeta,\lambda)d\zeta_R d\zeta_I, \tag{2.6}$$

$$g(z,\lambda) = -\left(\frac{1}{2\pi}\right)^2 \iint\limits_{\zeta\in\mathbb{C}} \frac{\exp(i/2(\zeta\bar z + \bar\zeta z))}{\zeta\bar\zeta + i(\lambda\bar\zeta + \zeta/\lambda)} d\zeta_R d\zeta_I, \tag{2.7}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$, $\zeta_R = \mathrm{Re}\zeta$, $\zeta_I = \mathrm{Im}\zeta$.

In terms of $\psi$ of (2.5) equation (2.6) takes the form

$$\psi(z,\lambda) = e^{-1/2(\lambda\bar{z}+z/\lambda)} + \iint\limits_{\zeta\in\mathbb{C}} G(z-\zeta,\lambda)v(\zeta)\psi(\zeta,\lambda)d\zeta_R d\zeta_I, \tag{2.8}$$

$$G(z,\lambda) = e^{-1/2(\lambda\bar{z}+z/\lambda)}g(z,\lambda), \tag{2.9}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$.

In terms of $m(z,\lambda) = (1+|z|)^{-(2+\varepsilon)/2}\mu(z,\lambda)$ equation (2.6) takes the form

$$m(z,\lambda) = (1+|z|)^{-(2+\varepsilon)/2} + \iint\limits_{\zeta\in\mathbb{C}} (1+|z|)^{-(2+\varepsilon)/2} g(z-\zeta,\lambda)\frac{v(\zeta)}{(1+|\zeta|)^{-(2+\varepsilon)/2}} m(\zeta,\lambda)d\zeta_R d\zeta_I, \tag{2.10}$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$. In addition, $A(\cdot,\cdot,\lambda) \in L^2(\mathbb{C}\times\mathbb{C})$, $|\mathrm{Tr}A^2(\lambda)| < \infty$, where $A(z,\zeta,\lambda)$ is the Schwartz kernel of the integral operator $A(\lambda)$ of the integral equation (2.10). Thus, the modified Fredholm determinant for (2.10) can be defined by means of the formula:

$$\ln\Delta(\lambda) = \mathrm{Tr}(\ln(I - A(\lambda)) + A(\lambda)) \tag{2.11}$$

(see [GK] for more precise sense of such definition).

Taking the subsequent members in the asymptotic expansion (2.5) for $\psi(z,\lambda)$, we obtain (see [N2]):

$$\psi(z,\lambda) = \exp\left(-\frac{1}{2}\left(\lambda\bar{z} + \frac{z}{\lambda}\right)\right)\Bigg\{1 - 2\pi\mathrm{sgn}(1-\lambda\bar{\lambda})\times$$
$$\times\left(\frac{i\lambda a(\lambda)}{z-\lambda^2\bar{z}} + \exp\left(-\frac{1}{2}\left(\left(\frac{1}{\bar{\lambda}}-\lambda\right)\bar{z} + \left(\frac{1}{\lambda}-\bar{\lambda}\right)z\right)\right)\frac{\bar{\lambda}b(\lambda)}{i(\bar{\lambda}^2 z - \bar{z})}\right) + o\left(\frac{1}{|z|}\right)\Bigg\}, \tag{2.12}$$

$|z| \to \infty$, $\lambda \in \mathbb{C}\backslash(\mathcal{E}\cup 0)$.

The functions $a(\lambda)$, $b(\lambda)$ from (2.12) are called the "scattering" data for the problem (2.1), (2.2) with $E = -1$. It is known that for $a(\lambda)$, $b(\lambda)$ the following formulas hold (see [N2]):

$$a(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{z\in\mathbb{C}} \mu(z,\lambda)v(z)dz_R dz_I, \tag{2.13}$$

$$b(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{z\in\mathbb{C}} \exp\left(-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar{\lambda}}\right)\bar{z} - \left(\bar{\lambda} - \frac{1}{\lambda}\right)z\right)\right)\mu(z,\lambda)v(z)dz_R dz_I, \tag{2.14}$$

where $\lambda \in \mathbb{C}\backslash(0 \cup \mathcal{E})$, $z_R = \mathrm{Re}z$, $z_I = \mathrm{Im}z$. In addition, formally, formulas (2.13), (2.14) can be written as

$$a(\lambda) = h(\lambda, \lambda), \quad b(\lambda) = h\left(\lambda, \frac{1}{\bar{\lambda}}\right), \tag{2.15}$$

where

$$h(\lambda, \lambda') = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{z\in\mathbb{C}} \exp\left(\frac{1}{2}\left(\lambda'\bar{z} + z/\lambda'\right)\right) \psi(z, \lambda) v(z) dz_R z_I, \tag{2.16}$$

and $\lambda \in \mathbb{C}\backslash(0 \cup \mathcal{E})$, $\lambda' \in \mathbb{C}\backslash 0$. (Note that, under assumptions (2.2), the integral in (2.16) is well–defined if $\lambda' = \lambda$ of if $\lambda' = 1/\bar{\lambda}$ but is not well–defined in general.)

Let

$$T = \{\lambda \in \mathbb{C}\colon |\lambda| = 1\}. \tag{2.17}$$

From (2.15), in particular, the following statement follows:

**Statement 2.1.** *Let (2.2) hold and $\Delta \neq 0$ on $T$. Then*

$$a(\lambda) = b(\lambda), \quad \lambda \in T. \tag{2.18}$$

The following properties of functions $\Delta(\lambda)$, $a(\lambda)$, $b(\lambda)$ will play a substantial role in the proof of Theorem 1.1.

**Statement 2.2.** *Let (2.2) hold. Then:*

1. $\Delta(\lambda) \in C(\mathbb{C})$;
2. $\Delta(\lambda) \to 1$ *as* $\lambda \to 0$ *or* $\lambda \to \infty$;
3. $\Delta(\lambda) \equiv \mathrm{const}$ *for* $\lambda \in T$;
4. $\Delta$ *is real–valued:* $\Delta = \bar{\Delta}$.
5. $\Delta(\lambda) = \Delta(1/\bar{\lambda})$, $\lambda \in \mathbb{C}\backslash 0$.

**Statement 2.3.** *Let conditions (2.2)–(2.3) be fulfilled. Then:*

- $\Delta(\lambda)$ *is a real–analytic function on* $D_+$, $D_-$, *where*

$$D_+ = \{\lambda \in \mathbb{C}\colon 0 < |\lambda| \leqslant 1\}, \quad D_- = \{\lambda \in \mathbb{C}\colon |\lambda| \geqslant 1\}. \tag{2.19}$$

- $a(\lambda) = \dfrac{\mathcal{A}(\lambda)}{\Delta(\lambda)}$, $b(\lambda) = \dfrac{\mathcal{B}(\lambda)}{\Delta(\lambda)}$, *where* $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$ *are real–analytic functions on* $D_+$, $D_-$.

Items 1–4 of Statement 2.2 are either known or follow from results mentioned in [HN], [N2] (see page 129 of [HN] and pages 420, 423, 429 of [N2]). In particular, item 1 of Statement 2.2 is a consequence of continuous dependency of $g(z,\lambda)$ on $\lambda \in \mathbb{C}\backslash 0$; item 3 of Statement 2.2 is a consequence of (2.11) and of the formula (see pages 420, 423 of [N2]) $G(z,\lambda) = (-i/4)H_0^1(i|z|)$, $z \in \mathbb{C}$, $\lambda \in T$, where $G$ is defined by (2.9), $H_0^1$ is the Hankel function of the first type. In addition, item 5 of Statement 2.2 follows from item 4 of this statement and from symmetry $\overline{G(z,\lambda)} = G(z, 1/\bar{\lambda})$, $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$.

Statement 2.3 is similar to Proposition 4.2 of [N2] and follow from: (i) formulas (2.13), (2.14), (ii) Cramer type formulas for solving the integral equation (2.10), (iii) the analog of Proposition 3.2 of [N2] for $g$ of (2.7).

Under assumptions (2.2), the function $\mu(z,\lambda)$, defined by (2.6), satisfies the following properties:

$$\mu(z,\lambda) \text{ is a continuous function of } \lambda \text{ on } \mathbb{C}\backslash(0 \cup \mathcal{E}); \tag{2.20}$$

$$\frac{\partial \mu(z,\lambda)}{\partial \bar{\lambda}} = r(z,\lambda)\overline{\mu(z,\lambda)}, \tag{2.21a}$$

$$r(z,\lambda) = r(\lambda) \exp\left(\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar{\lambda}}\right)\bar{z} - \left(\bar{\lambda} - \frac{1}{\lambda}\right)z\right)\right), \tag{2.21b}$$

$$r(\lambda) = \frac{\pi \operatorname{sgn}(1-\lambda\bar{\lambda})}{\bar{\lambda}} b(\lambda) \tag{2.21c}$$

for $\lambda \in \mathbb{C}\backslash(0 \cup \mathcal{E})$;

$$\mu \to 1, \text{ as } \lambda \to \infty,\ \lambda \to 0. \tag{2.22}$$

The function $b$ possesses the following properties (see [GN2], [N2]):

$$b \in C(\mathbb{C}\backslash\mathcal{E}), \tag{2.23}$$

$$b\left(-\frac{1}{\bar{\lambda}}\right) = b(\lambda), \quad b\left(\frac{1}{\bar{\lambda}}\right) = \overline{b(\lambda)}, \quad \lambda \in \mathbb{C}\backslash 0, \tag{2.24}$$

$$\lambda^{-1}b(\lambda) \in L_p(D_+) \text{ (as a function of } \lambda) \text{ if } \mathcal{E} = \varnothing, \quad 2 < p < 4. \tag{2.25}$$

In addition, the following theorem is valid:

**Theorem 2.1** ([GN2], [N2]). *Let $v$ satisfy (2.2) and $\mathcal{E} = \varnothing$ for this potential. Then $v$ is uniquely determined by its scattering data $b$ (by means of (2.20), (2.21) and equation (1.1) for $E = -1$ and $\psi$ of (2.5) ).*

Finally, if $(v(z,t), w(z,t))$ is a solution of equation (1.1) with $E = -1$, where $(v(z,t), w(z,t))$ satisfy the following conditions:

$v, w \in C(\mathbb{R}^2 \times \mathbb{R})$ and for each $t \in \mathbb{R}$ the following properties are fulfilled:

$$v(\cdot, t) \in C^3(\mathbb{R}^2), \quad \partial_x^j v(x,t) = O\left(|x|^{-2-\varepsilon}\right) \text{ for } |x| \to \infty, |j| \leq 3 \text{ and some } \varepsilon > 0, \tag{2.26}$$

$w(x,t) \to 0$ for $|x| \to \infty$,

then the dynamics of the scattering data is described by the following equations

$$a(\lambda, t) = a(\lambda, 0), \tag{2.27}$$

$$b(\lambda, t) = \exp\left\{\left(\lambda^3 + \frac{1}{\lambda^3} - \bar{\lambda}^3 - \frac{1}{\bar{\lambda}^3}\right) t\right\} b(\lambda, 0), \tag{2.28}$$

where $\lambda \in \mathbb{C}\backslash 0$, $t \in \mathbb{R}$.

# 3 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on Proposition 3.1 and Proposition 3.2 given below.

**Lemma 3.1.** *Let $v(z)$ satisfy (2.2) and $a(\lambda)$, $b(\lambda)$ be the scattering data corresponding to $v(z)$. Then the scattering data $a_\zeta(\lambda)$, $b_\zeta(\lambda)$ for the potential $v_\zeta(z) = v(z-\zeta)$ are related to $a(\lambda)$, $b(\lambda)$ by the formulas*

$$a_\zeta(\lambda) = a(\lambda), \tag{3.1}$$

$$b_\zeta(\lambda) = \exp\left(-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar{\lambda}}\right)\bar{\zeta} - \left(\bar{\lambda} - \frac{1}{\lambda}\right)\zeta\right)\right) b(\lambda), \tag{3.2}$$

*where $z, \zeta \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$.*

*Proof.* We first note that $\psi(z-\zeta, \lambda)$ satisfies equation (2.1) with the operator $L = -\Delta + v_\zeta(z)$. Then the function $\psi_\zeta(z, \lambda)$ corresponding to $v_\zeta(z)$ and possessing the asymptotics (2.5) is $\psi_\zeta(z, \lambda) = e^{-\frac{1}{2}\left(\lambda\bar{\zeta} + \zeta/\lambda\right)}\psi(z-\zeta, \lambda)$. In terms of function $\mu$ this relation is written $\mu_\zeta(z, \lambda) = \mu(z-\zeta, \lambda)$ Thus we have

$$a_\zeta(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{z \in \mathbb{C}} v(z-\zeta)\mu(z-\zeta, \lambda) dz_R dz_I = a(\lambda),$$

and, similarly,

$$b_\zeta(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{z\in\mathbb{C}} \exp\left(-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar\lambda}\right)\bar z - \left(\bar\lambda - \frac{1}{\lambda}\right) z\right)\right) \times$$

$$\times\, v(z-\zeta)\mu(z-\zeta,\lambda)dz_R dz_I =$$

$$= \exp\left(-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar\lambda}\right)\bar\zeta - \left(\bar\lambda - \frac{1}{\lambda}\right)\zeta\right)\right) b(\lambda).$$

□

**Proposition 3.1.** *Let $(v(z,t), w(z,t))$ be an exponentially localized soliton of (1.1) in the sense (1.7). Let $b(\lambda,t)$ be the scattering data for $v(z,t)$ for some $E = E_{fixed} < 0$. Then $b(\lambda,t) \equiv 0$ in the domain where it is well–defined, i.e. in $\mathbb{C}\backslash\mathcal{E}$, where $\mathcal{E}$ is defined by (2.4).*

*Proof.* In virtue of (2.28) and Statement 2.3 it is sufficient to prove that $b(\lambda,0)\equiv 0$ in some neighborhoods of 0 and $\infty$.

Let $U_0$, $U_\infty$ be the neighborhoods of 0 and $\infty$, respectively, such that $\Delta \neq 0$ in $U_0$, $U_\infty$ (such neighborhoods exist in virtue of item 2 of Statement 2.2). For $\lambda \in U_0 \cup U_\infty$ the function $b(\lambda,0)$ is well–defined and continuous. As $(v(z,t), w(z,t))$ is a soliton, the dynamics of the function $b(\lambda,t)$ can be written as

$$b(\lambda,t) = \exp\left(-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar\lambda}\right)\bar c - \left(\bar\lambda - \frac{1}{\lambda}\right)c\right)t\right) b(\lambda,0) \qquad (3.3)$$

(see Lemma 3.1).

Combining this with formula (2.28), we obtain

$$\exp\left\{-\frac{1}{2}\left(\left(\lambda - \frac{1}{\bar\lambda}\right)\bar c - \left(\bar\lambda - \frac{1}{\lambda}\right)c\right)t\right\} b(\lambda,0) =$$

$$= \exp\left\{\left(\lambda^3 + \frac{1}{\lambda^3} - \bar\lambda^3 - \frac{1}{\bar\lambda^3}\right)t\right\} b(\lambda,0).$$

As functions $\lambda$, $\bar\lambda$, $\frac{1}{\lambda}$, $\frac{1}{\bar\lambda}$, $\lambda^3$, $\bar\lambda^3$, $\frac{1}{\lambda^3}$, $\frac{1}{\bar\lambda^3}$, 1 are linearly independent in any neighborhood of 0 and $\infty$, we obtain that $b(\lambda,0)\equiv 0$ in $U_0\cup U_\infty$. □

**Proposition 3.2.** *Let $v(z)$ satisfy (2.2)–(2.3) and $b(\lambda)$ be its scattering data for some $E = E_{fixed} < 0$. If $b(\lambda)\equiv 0$ in the domain where it is well–defined, i.e. in $\mathbb{C}\backslash\mathcal{E}$, where $\mathcal{E}$ is defined by (2.4), then $v\equiv 0$.*

Note that Proposition 3.2 can be considered as an analog of Corollary 3 of [GN1].

*Proof of Proposition 3.2.* 1. First we will prove that from the assumptions of this proposition it follows that $a(\lambda) \equiv 0$ in $U_0 \cup U_\infty$, where $U_0$ and $U_\infty$ are such neighborhoods of 0 and $\infty$, respectively, that $\Delta(\lambda) \neq 0$ for $\lambda \in U_0 \cup U_\infty$. We note that from (2.13), (2.14), (2.21a), (2.22) it follows that

$$\frac{\partial a(\lambda)}{\partial \bar{\lambda}} = \frac{\pi \mathrm{sgn}(1 - \lambda\bar{\lambda})}{\bar{\lambda}} b(\lambda)\overline{b(\lambda)}, \quad \lambda \in \mathbb{C}\backslash(\mathcal{E} \cup 0), \tag{3.4}$$

$$a(\lambda) \to \hat{v}(0) \text{ as } \lambda \to \infty \text{ or } \lambda \to 0, \text{ where} \tag{3.5}$$

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{z\in\mathbb{C}} e^{\frac{i}{2}(\bar{p}z + p\bar{z})} v(z) dz_R dz_I, \quad p \in \mathbb{C}. \tag{3.6}$$

It means that

$$a(\lambda) \text{ is holomorphic in } \mathbb{C}\backslash(\mathcal{E}). \tag{3.7}$$

According to item 3 of Statement 2.2, $\Delta(\lambda) \equiv \mathrm{const}$ for $\lambda \in T$. We will consider separately two cases: $\Delta \equiv C \neq 0$ on $T$ and $\Delta \equiv 0$ on $T$.

(a) $\Delta(\lambda) \equiv C \neq 0$ on $T$:
From item 1 of Statement 2.2 it follows that there exists $U_T$, a neighborhood of $T$, such that $\Delta(\lambda) \neq 0$ in $U_T$. Thus $a(\lambda)$ is holomorphic in $U_T$. From Statement 2.1 we obtain that $a(\lambda) = b(\lambda) = 0$ on $T$. It follows then that $a(\lambda) \equiv 0$ in $U_T$. Using statement 2.3, we obtain that $a(\lambda) \equiv 0$ in $U_0 \cup U_\infty$.

(b) $\Delta(\lambda) \equiv 0$ on $T$:
In [HN] the $\bar{\partial}$–equation for $\Delta$ was derived. In variables $\lambda$, $\bar{\lambda}$ it is written as

$$\frac{\partial \ln \Delta(\lambda)}{\partial \bar{\lambda}} = -\frac{\pi \mathrm{sgn}(\lambda\bar{\lambda} - 1)}{\bar{\lambda}} \left( a\left(\frac{1}{\bar{\lambda}}\right) - \hat{v}(0) \right). \tag{3.8}$$

Equation (3.8) and properties (3.5), (3.7) imply that $\frac{\partial \ln \Delta}{\partial \lambda}$ is an antiholomorphic function in $U_0 \cup U_\infty$, where $\Delta$ is close to 1 and, thus, $\ln \Delta$ is a well–defined one–valued function. As $\Delta$ is a real–valued real analytic function, it follows that

$$\ln \Delta = f(\lambda) + \overline{f(\lambda)} \tag{3.9}$$

for some holomorphic function $f(\lambda)$, or

$$\Delta = F(\lambda)\overline{F(\lambda)} \tag{3.10}$$

for some holomorphic function $F(\lambda)$ on $U_0 \cup U_\infty$. Now we will use the following lemma (the proof of this lemma is given in Section 4):

**Lemma 3.2.** *Let $\Delta(\lambda)$ be real–analytic in $D_+ = \{\lambda \in \mathbb{C}\colon 0 < |\lambda| \le 1\}$. Suppose that $\Delta(\lambda)$ can be represented as*

$$\Delta(\lambda) = F(\lambda)\overline{F(\lambda)}, \quad \lambda \in U_0, \tag{3.11}$$

*for some function $F(\lambda)$ holomorphic on $U_0$, a neighborhood of zero. Then the representation (3.11) holds on $D_+$, i.e. $F(\lambda)$ can be extended analytically to $D_+$.*

Thus, the representation (3.10) is valid separately on $D_+$ and on $D_-$, where we used also item 5 of Statement 2.2. As $\Delta(\lambda) \equiv 0$ on $T$, it follows that $F(\lambda) \equiv 0$ on $T$ and, further, $F(\lambda) \equiv 0$ on $\mathbb{C}$. This contradicts with item 2 of Statement 2.2. Thus we have shown that under the assumptions of Proposition 3.2 the case $\Delta(\lambda) \equiv 0$ on $T$ cannot hold.

2. Our next step is to prove that $\Delta(\lambda) \equiv 1$ for $\lambda \in \mathbb{C}$.

   Formula (3.5) states that $a(0) = a(\infty) = \hat{v}(0)$. Thus from equation (3.8) it follows that $\frac{\partial \ln \Delta}{\partial \bar{\lambda}} = 0$, and $\ln \Delta$ is holomorphic in some neighborhood of 0 and $\infty$. As $\Delta(\lambda)$ is a real–valued function and item 2 of Statement 2.2 holds, we conclude that $\Delta \equiv 1$ in some neighborhood of 0 and $\infty$. Now using Statement 2.3, we obtain that $\Delta \equiv 1$ in $\mathbb{C}$ and, as a corollary, $\mathcal{E} = \varnothing$.

3. From the previous item it follows that equation (2.21a) holds for $\forall \lambda$: $\lambda \in \mathbb{C} \backslash 0$. Due to the assumptions of Proposition 3.2 and the property that $\mathcal{E} = \varnothing$, we have that $b \equiv 0$ on $\mathbb{C}$ which means that $\mu(z, \lambda)$ is holomorphic on $D_+$, $D_-$. As it is also continuous on $\mathbb{C}$ and property (2.22) holds, we conclude that $\mu(z, \lambda) \equiv 1$ and $v(z) \equiv 0$.

□

*Proof of Theorem 1.1 for $E = E_{fixed} < 0$.* The result follows immediately from Propositions 3.1, 3.2. □

# 4 Proof of Lemma 3.2

As $F(\lambda)$ is analytic in $U_0$, it can be represented in this domain by a Taylor series. Let us consider the radius of convergence $R$ of this Taylor series. Suppose that the statement of Lemma 3.2 is not true and $R < 1$.

Let us take a point $\lambda_0$, such that $|\lambda_0| = R$. In this point $\Delta(\lambda)$ can be represented by the following series

$$\Delta(\lambda) = \sum_{k,j=0}^{\infty} b_{k,j}(\lambda - \lambda_0)^k(\bar{\lambda} - \bar{\lambda}_0)^j \tag{4.1}$$

uniformly convergent in $U_{\lambda_0}$, some neighborhood of $\lambda_0$. We will prove that the coefficients $b_{j,k}$ satisfy the following properties:

$$\begin{aligned} &b_{k,k} \in \mathbb{R}, \quad b_{k,k} \geq 0; && (a)\\ &b_{k,j} = \overline{b_{j,k}}; && (b)\\ &b_{k,j}b_{m,l} = b_{k,l}b_{m,j}. && (c) \end{aligned}$$

Indeed,

$$(a)\colon b_{k,k} = \frac{1}{(k!)^2}\partial_\lambda^k \partial_{\bar{\lambda}}^k \Delta(\lambda)\Big|_{\lambda=\lambda_0} = \lim_{\lambda\to\lambda_0}\frac{1}{(k!)^2}\partial_\lambda^k \partial_{\bar{\lambda}}^k \Delta(\lambda) = \lim_{\lambda\to\lambda_0}\frac{1}{(k!)^2}|\partial_\lambda^k F(\lambda)|^2 \in \mathbb{R}, \geq 0.$$

$$(b)\colon b_{k,j} = \lim_{\lambda\to\lambda_0}\frac{1}{k!j!}\partial_\lambda^k F(\lambda)\overline{\partial_\lambda^j F(\lambda)} = \lim_{\lambda\to\lambda_0}\overline{\frac{1}{k!j!}\overline{\partial_\lambda^k F(\lambda)}\partial_\lambda^j F(\lambda)} = \overline{b_{j,k}}.$$

$$(c)\colon b_{k,j}b_{m,l} = \lim_{\lambda\to\lambda_0}\frac{1}{k!j!m!l!}\partial_\lambda^k F(\lambda)\overline{\partial_\lambda^j F(\lambda)}\partial_\lambda^m F(\lambda)\overline{\partial_\lambda^l F(\lambda)} = b_{k,l}b_{m,j}.$$

From properties (a)–(c) it follows that there exist such $a_k \in \mathbb{C}$, $k = 0, 1, \ldots$, that

$$b_{k,j} = a_k\bar{a}_j. \tag{4.2}$$

We will prove this statement by considering two different cases:

1. $b_{k,k} = 0\ \forall k \in \mathbb{N} \cup 0$

   In this case from properties (b), (c) it follows that $b_{k,j} = 0\ \forall k, j \in \mathbb{N} \cup 0$, and we can take $a_k = 0\ \forall k \in \mathbb{N} \cup 0$.

2. $b_{k,k} \neq 0$ for some $k \in \mathbb{N} \cup 0$.

   In this case we take $l$ to be the minimal number such that $b_{l,l} \neq 0$. Then we set $a_0 = a_1 = \ldots = a_{l-1} = 0$ and we take an arbitrary complex number $a_l$ satisfying $|a_l|^2 = b_{l,l}$. For the rest of the coefficients we set

$$a_n = \frac{b_{n,l}}{\bar{a}_l}, \tag{4.3}$$

where $n = l+1, l+2, \ldots$.

Now let us prove property (4.2). Let us suppose that $k < l$. Then $a_k = 0$, $b_{k,k} = 0$ and from properties (b), (c) it follows that $b_{k,j} = 0$ $\forall j \in \mathbb{N} \cup 0$. Thus property (4.2) holds when $k < l$. A similar reasoning can be carried out when $j < l$. Now let us suppose that $k \geq l$, $j \geq l$. Then

$$a_k \bar{a}_j = \frac{b_{k,l}\bar{b}_{j,l}}{\bar{a}_l a_l} = \frac{b_{k,l} b_{l,j}}{b_{l,l}} = b_{k,j}. \tag{4.4}$$

Representation (4.2) is proved.

From convergence of series (4.1) it follows that the following series

$$F_1(\lambda) = \sum_{k=0}^{\infty} a_k (\lambda - \lambda_0)^k \tag{4.5}$$

converges uniformly in $U_{\lambda_0}$ (indeed, the case when $b_{k,k} = 0$ $\forall k \in \mathbb{N} \cup 0$ is trivial, and in the case when $\exists l\colon b_{l,l} \neq 0$ we take the sum of the members of series (4.1) with coefficients $b_{k,l}$, $k = 0, 1, \ldots$, and obtain series (4.5) multiplied by $\bar{a}_l(\bar{\lambda} - \bar{\lambda}_0)^l$). Thus there exists the function $F_1(\lambda)$ analytic in $U_0$ such that $\Delta(\lambda) = F_1(\lambda)\overline{F_1(\lambda)}$. Consequently, we have two functions $F(\lambda)$ and $F_1(\lambda)$ analytic in a common domain lying in $\{\lambda \in \mathbb{C}\colon |\lambda| \leq R\} \cap U_{\lambda_0}$ and such that $|F(\lambda)| = |F_1(\lambda)|$. It means that $F(\lambda)$ and $F_1(\lambda)$ are equal up to a constant factor: $F(\lambda) = \mu F_1(\lambda)$, $|\mu| = 1$. It follows then that $\mu F_1(\lambda)$ is an analytic continuation of $F(\lambda)$ to $U_{\lambda_0}$.

The same reasoning can be applied to any point $\lambda_0$ on the boundary of the ball $B_R = \{\lambda \in \mathbb{C}\colon |\lambda| \leq R\}$, i.e. $F(\lambda)$ can be continued analytically to some larger domain. Hence we obtain a contradiction to the assumption that $R < 1$ is the radius of convergence of the Taylor series for $F(\lambda)$. Thus $R = 1$ and $F(\lambda)$ can be extended analytically to $D_+$. $\square$

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